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LETTER TO THE EDITOR

On solutions of the equation $\nabla \times \boldsymbol{a} = k\boldsymbol{a}$

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Abstract. The vector differential equation $\nabla \times \boldsymbol{a} = k\boldsymbol{a}$ with k constant appears in models of 'force-free magnetic fields' in plasma physics and in astrophysics, and also in a model for 'force-free electromagnetic waves.' Solutions of this equation which are identified with physical fields exhibit several inconsistencies. This letter identifies the source of these inconsistencies as the lack of covariance of this equation with respect to transformations.

The first-order vector differential equation,

$$\nabla \times \boldsymbol{a} = k\boldsymbol{a} \qquad k = \text{constant} \tag{1}$$

has in the past been proposed as a field equation in a static model of 'force-free magnetic fields' (Ferraro and Plumpton 1961, Freire 1966, Moffatt 1978, Roberts 1967). Explicit solutions exist, but it is not entirely clear, however, if they correspond to actual magnetic fields (Akasofu and Chapman 1972, Aly 1984, Cowling 1976, Low 1977, 1982, Svestka 1976). It is not our intention to discuss the validity of equation (1) in any particular physical model, but to point out several mathematical peculiarities which are basic features of its solutions (Salingaros and Saflekos 1986).

In a separate but related development, a model for 'force-free electromagnetic waves' was based on equation (1) (Chu and Ohkawa 1982). This proposal was criticised on physical grounds by Lee (1983), and by this author (Salingaros 1985). This letter traces the origin of the inconsistencies present in solutions of equation (1) which have been used in physical models to the mathematical properties of the equation itself.

Arguments are presented which demonstrate that equation (1) has solutions which may not, however, be identified with physical magnetic fields. This result is based on the transformation properties of equation (1) under (i) gauge transformations, (ii) parity and (iii) a change of basis.

First, consider a gauge transformation of the electromagnetic vector potential $\boldsymbol{A}$. Any physical magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ must be invariant under the addition of the gradient of an arbitrary scalar function λ to $\boldsymbol{A}$ (Landau and Lifshitz 1975)

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\lambda \Rightarrow \boldsymbol{B}' = \boldsymbol{B}. \tag{2}$$

If a solution of equation (1) is identified with a magnetic field, then that magnetic field would be gauge-dependent. This follows from the ability to relate the vector $\boldsymbol{B}$ directly to its vector potential $\boldsymbol{A}$ via some scalar function ϕ independent of λ. One has, from equations (1) and (2)

$$\nabla \times \boldsymbol{B} = k\boldsymbol{B} = k\nabla \times \boldsymbol{A} \Rightarrow \boldsymbol{B} = k\boldsymbol{A} + \nabla\phi \tag{3a}$$

$$\boldsymbol{B}' = k\boldsymbol{A}' + \nabla\phi = \boldsymbol{B} + k\nabla\lambda \qquad \nabla \cdot \boldsymbol{B}' = k\nabla^2\lambda \neq 0. \tag{3b}$$

Therefore, a solution of equation (1) which is required to be solenoidal and gauge-invariant will exist only for $k = 0$, or in the specific gauge $\lambda = 0$. The assumption that equation (1) is an equation for a magnetic field leads to a violation of gauge invariance.

The second invariance to be considered is reflection invariance, or invariance under parity. In this case one can *a priori* rule out solutions of equation (1) which have definite parity. If a solution $\boldsymbol{a}$ is an eigenvector of parity, then equation (1) is not invariant under parity, since the gradient ∇ is a vector operator which changes sign upon reflection of the cartesian coordinates, $\nabla' = -\nabla$. If one requires that $\boldsymbol{a}' = -\boldsymbol{a}$, or $\boldsymbol{a}' = \boldsymbol{a}$, then $\boldsymbol{a} = 0$ identically. It follows, therefore, that any solution of equation (1) must necessarily not be an eigenvector of parity (Lee 1983, Salingaros and Saflekos 1986).

A magnetic field $\boldsymbol{B}$ is known to be a parity eigenvector of even parity, as $\boldsymbol{B}$ is an axial vector or pseudovector (Landau and Lifshitz 1975). This characteristic implies that a magnetic field cannot be a solution of equation (1), independently from the gauge invariance.

Following the above discussion, actual solutions of equation (1) will have mixed parity. It may be that each component of $\boldsymbol{a}$ in cartesian coordinates will have indefinite parity, but what is more interesting, is where each component of $\boldsymbol{a}$ is a parity eigenstate. Without loss of generality, it is sufficient to assume that $\boldsymbol{a}$ possesses one odd and two even functions as components, $a_i'(\boldsymbol{r}) = a_i(-\boldsymbol{r})$:

$$\boldsymbol{a}' = (a_1', a_2', a_3') = (-a_1, a_2, a_3). \tag{4}$$

The form invariance of equation (1), and also of the divergence condition $\nabla \cdot \boldsymbol{a} = 0$ which follows from equation (1), imposes extra conditions on the components of the solution (4). One has altogether eight conditions arising from parity covariance in general;

$$\begin{aligned} &\nabla \times \boldsymbol{a} = k\boldsymbol{a} \qquad && \nabla \cdot \boldsymbol{a} = 0 \\ &-\nabla \times \boldsymbol{a}' = k\boldsymbol{a}' \qquad && \nabla \cdot \boldsymbol{a}' = 0. \end{aligned} \tag{5}$$

It is easy to see that equations (4) and (5) reduce to just four conditions, and that the components of $\boldsymbol{a}$ are functions of the coordinates x^2 and x^3 only. The problem therefore is fixed by the parity to a two-dimensional variable space

$$\begin{aligned} &\partial_2 a_2 + \partial_3 a_3 = 0 \\ &\partial_2 a_3 - \partial_3 a_2 = k a_1 \\ &\partial_3 a_1 = k a_2 \qquad \partial_2 a_1 = -k a_3. \end{aligned} \tag{6}$$

These equations (6) together imply that the a_i satisfy the two-dimensional wave equation

$$(\partial_2^2 + \partial_3^2 + k^2) a_i = 0 \qquad i = 1, 2, 3. \tag{7}$$

(By contrast, the wave equation in all three variables follows from taking the curl of equation (1).)

It is therefore sufficient to find any function $a_1(x^2, x^3)$ which by definition (4) is odd (pseudoscalar) and which is a solution of equation (7). Then, the other two components of the solution to equation (1) follow from (6) as the two partial derivatives. One has a general form for the solution which transforms under parity according to (4) in terms of a single pseudoscalar function $\psi(x^2, x^3)$, in cartesian coordinates

$$\boldsymbol{a}(x^2, x^3) = [\psi, (1/k)\partial_3\psi, -(1/k)\partial_2\psi] \tag{8a}$$

$$\psi(-x^2, -x^3) = -\psi(x^2, x^3) \qquad (\partial_2^2 + \partial_3^2 + k^2)\psi = 0. \tag{8b}$$

This general form of the solution is clearly independent of the specific choice made in (4); one may permute (8a) to describe other solutions whose components are parity eigenstates in the cartesian coordinate system. If instead two pseudoscalars are chosen in (4), then the generating function ψ will be a true scalar function, and its partial derivatives will be the two pseudoscalar components.

The standard example of a solution to equation (1) in cartesian coordinates is a special case of equation (8) (Ferraro and Plumpton 1961, Freire 1966, Low 1982, Moffatt 1978). Here, ψ is a function of one variable $z = x^3$ only;

$$\psi(z) = a \sin kz \Rightarrow \boldsymbol{a}(z) = a(\sin kz, \cos kz, 0). \tag{9}$$

The above analysis illustrates that solutions to equation (1) are of a very specific form. A consequence of this special form is the general lack of covariance under a change of basis. Consider a transformation of the coordinates

$$(x^2, x^3) \rightarrow (u(x^2, x^3), v(x^2, x^3)). \tag{10}$$

Then, the solution (8a) changes form so that it no longer satisfies the original equations (5). One has

$$\boldsymbol{a}'(u, v) = \left(\psi, \frac{J}{k}\frac{\partial\psi}{\partial v}, -\frac{J}{k}\frac{\partial\psi}{\partial u}\right)^{-1}. \tag{11}$$

The solution would be covariant only if

$$\boldsymbol{a}'(u, v) = \left(\psi, \frac{1}{k}\frac{\partial\psi}{\partial v}, -\frac{1}{k}\frac{\partial\psi}{\partial u}\right)' \tag{12}$$

but that is the case only when the Jacobian $J = \partial_2 u \partial_3 v - \partial_3 u \partial_2 v$ equals $+1$, i.e. for translations and rotations only. The solution is not form-invariant under any other transformation of the coordinates. Even though this analysis was based upon the specific form (4), it is true in general.

The independence of the choice of reference system is essential for all differential equations which describe physical fields (Gelfand *et al* 1963).

The results of this discussion may be summarised as follows: one can find a special explicit solution of equation (1) in a specific basis or coordinate system. Such a solution does not, however, transform the way a magnetic field should transform. These peculiar mathematical characteristics question the use of equation (1) in models of electric or magnetic fields. It is felt that these results are relevant in view of the extensive applications of this equation in models of magnetic fields in both plasma physics and in astrophysics.

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